# Multi-parameter exponentially fitted, P-stable Obrechkoff methods

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**Abstract.** We consider the construction of P-stable, multi-parameter exponentially fitted Obrechkoff methods for second order differential equations. An earlier result for single-parameter exponential fitting is re-examined and extended to multi-parameter, multi-order exponential fitting.

**Keywords:** P-stability, Exponential fitting, Obrechkoff, Rational approximation, Multi-parameter
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## INTRODUCTION

In [1], the authors considered symmetric two-step Obrechkoff methods of the form

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{j=1}^{m} h^{2j} \left[\beta_{j0} y_{n+1}^{(2j)} + 2\beta_{j1} y_n^{(2j)} + \beta_{j0} y_{n-1}^{(2j)}\right] \tag{1}$$

and showed that P-stability, a desirable property when dealing with periodic problems, can be guaranteed by imposing that the stability function $R_{mm}(v^2)$ of the Obrechkoff method equals $Re\left(\hat{P}_m^m(iv)\right)$, the real part of the $[m/m]$-Padé approximation of $e^{iv}$.

Extending this result in [2], the technique of exponential fitting [3] is brought into the picture. Whereas a classical Obrechkoff method of order $2p$ has a polynomial fitting space

$$\{1, t, t^2, \ldots, t^{2p+1}\}$$

(i.e. the method is exact for solutions that fall within that space), an exponentially fitted variant can have a so called mixed-type fitting space (for any $K > 0$ and $P < p$ such that $K + 2P = 2p - 1$)

$$\{1, t, t^2, \ldots, t^K, e^{\pm\mu_0 t}, te^{\pm\mu_0 t}, \ldots, t^P e^{\pm\mu_0 t}\} \tag{2}$$

that contains one free parameter $\mu_0$. If $i\mu_0 \in \mathbb{R}$, this technique is also called trigonometric fitting because the fitting space then contains pairs of sine and cosine. A more general approach often called multi-parameter exponential fitting, uses fitting spaces of the form

$$\{1, t, t^2, \ldots, t^K, e^{\pm\mu_0 t}, e^{\pm\mu_1 t}, \ldots, e^{\pm\mu_P t}\}. \tag{3}$$

and it is known that (2) is recovered when each $\mu_j$ approaches $\mu_0$. In turn, the polynomial fitting space is found when $\mu_0 \to 0$. For this family of methods, basis functions $y(t) = 1$ and $y(t) = t$ are always included in the fitting space due to consistency and symmetry.

It was shown that P-stability can also be guaranteed for exponentially fitted Obrechkoff methods of type (2) with $i\mu \in \mathbb{R}$. This can be achieved by imposing that the stability function $R_{mm}(v^2)$ is the real part of an EF $[m/m]$-Padé approximation ${}^*\hat{P}_m^m(v)$ of $e^{iv}$. This approximant is of the form

$${}^*\hat{P}_m^m(v) = \frac{V_m(v)}{V_m(-v)}, \quad V_m(v) = 1 + \sum_{j=1}^{m} a_j(\mu) v^j$$

and can be constructed by imposing

$$\begin{cases} \dfrac{\partial^q}{\partial x^q} F(x,t)|_{(x,t)=(0,\mu)} = 0 \quad q = 1\dots K & \text{(4a)} \\ Re\left(\dfrac{\partial^q}{\partial t^q} F(x,t)|_{(x,t)=(i,\mu)}\right) = 0 \quad q = 0\dots P & \text{(4b)} \end{cases}$$

on

$$F(x,t) := e^{tx} V_m(-v) - V_m(v).$$

In this paper, we want to show how this result can be simplified and how it extends to multi-parameter exponential fitting.

# DERIVATION

## Classical methods

The linear stability theory for this type of methods (i.e. for second order differential equations) was introduced in [4] and is related to the test equation

$$y'' = -k^2 y, k \in \mathbb{R}$$

with general solution $y(t) = c_1 \cos(kt) + c_2 \sin(kt)$. With this test equation, one obtains from (1)

$$y_{n+1} - 2R_{mm}(v^2) y_n + y_{n-1} = 0 \tag{5}$$

with $v := kh$ and $R_{mm}(v^2)$ a rational function of order $m/m$. Idealiter, (5) is satisfied exactly for all solutions of the test equation, which would require $R_{mm}(v^2) \equiv \cos(v)$.

It seems obvious to choose the coefficients of the method such that $R_{mm}(v^2)$ is the $[m/m]$-Padé approximation of $\cos(v)$. While this leads to methods with maximal order $4m$, it does not provide P-stability. To obtain that, as shown in [1], the approximation $\cos(v) \approx Re\left(\hat{P}_m^m(iv)\right)$ is better suited due to its boundedness on the entire real axis. It can be shown that

$$Re\left(\hat{P}_m^m(iv)\right) = \frac{1}{2}\left[\hat{P}_m^m(iv) + \hat{P}_m^m(-iv)\right], \tag{6}$$

in which one recognizes the exponential form of $\cos(v)$. The property that $R_{mm}(v^2)$ is a rational function in $v^2$ allows to decompose it as

$$R_{mm}(v^2) = \frac{1}{2}\left[S_{mm}(iv) + S_{mm}(-iv)\right], \tag{7}$$

with $S_{mm}(v)$ also a rational function of degree $m/m$. Combining this with (6) leads to the requirement that $S_{mm}(iv) = \hat{P}_m^m(iv)$. From the properties of Padé approximations, the rules to impose on the stability function become

$$S_{mm}^{(j)}(0) = 1, \quad j = 0\dots 2m.$$

This set of conditions is equivalent to (4) in case $K = 2m+1$.

## Exponentially fitted methods

Suppose we want to construct an Obrechkoff method with a fully-tuned multi-parameter fitting space

$$\{e^{\mu_1 t}, e^{\mu_2 t}, \dots, e^{\mu_{2p} t}\},$$

assuming distinct values for all $\mu_j$. This can only be accomplished by imposing that (5) is fulfilled for $v = iw_j$, $w_j := i\mu_j h$, $j = 1 \dots 2p$. The stability function still has the same structure, so we can decompose $R_{mm}(v)$ as (7) again and obtain

$$\begin{cases} S_{mm}(w_j) = e^{w_j}, \quad j = 1 \dots 2p & \text{(8a)} \\ S_{mm}(-w_j) = e^{-w_j}, \quad j = 1 \dots 2p. & \text{(8b)} \end{cases}$$

This time, there is no obvious symmetry, so both terms contribute $2p$ conditions. Since $S_{mm}$ is an expression with only $2m$ unknowns, $p$ can be at most $\lfloor m/2 \rfloor$ if no conditions coincide. When all these conditions are satisfied, $R_{mm}(v)$ is a rational interpolant of $\cos(v)$ through $v = \{v_1, \dots, v_{2p}\}$.

In [5], the same conditions are applied directly to $R_{mm}(v^2)$ and $\cos(v)$. This approach, however, does not necessarily produce P-stable methods.

### *Symmetric exponential fitting*

Since we are considering a family of methods for second order differential equations, it is more sensible to talk about trigonometric fitting with symmetric fitting spaces of type (2) or (3), instead of asymmetrical exponential fitting as above. It is known that the latter approach can be transformed into trigonometric fitting by taking

$$\mu_{p+j} = -\mu_j, \quad j = 1 \dots p.$$

This step folds the two parts of (8) halfway onto each other, reducing the number of conditions to $2p$ and thus allowing $m$ pairs of exponentials in the fitting space. Under these conditions, the interpolation difference function $E(w) := S_{mm}(w) - e^w$ has $p$ pairs of roots, each symmetrical across the origin.

### *Mixed-type, single-parameter EF*

The conditions to impose on the stability function of a method with fitting space of the form (2) with $K + 2P = 2p - 1$ can be found by taking an additional step

$$\begin{aligned} \mu_r &\to 0, \quad j = 0 \dots K \\ \mu_j &\to \mu_0, \quad j = K+1 \dots K+1+P. \end{aligned}$$

This causes the roots of $E(w)$ to converge at $w = 0$ and $w = \pm h\mu_0$, leading to

$$\begin{cases} S_{mm}^{(j)}(0) = 1, \quad j = 0 \dots K \\ S_{mm}^{(j)}(h\mu_0) = e^{h\mu_0}, \quad j = 0 \dots P \\ S_{mm}^{(j)}(-h\mu_0) = e^{-h\mu_0}, \quad j = 0 \dots P. \end{cases}$$

Under these conditions, $R_{mm}(v)$ becomes multi-point Padé approximation to $\cos(v)$ of order $K$ in $v = 0$ and of order $P$ in $v = ih\mu$ and $v = -ih\mu$. It can be shown that when $\mu_0$ is real, these condition coincide with (4).

For example, the stability function of an Obrechkoff method with $m = 2$ and fitting space $\{1, t, t^2, t^3, e^{\pm\mu t}\}$ can be built with

$$S_{22}(v) = -\frac{((\mu - 2)\,e^{\mu} + 2 + \mu)\,v^2 + \left(-\mu^2 + e^{\mu}\mu^2\right)v + 2\,e^{\mu}\mu^2 - 2\,\mu^2}{((-\mu + 2)\,e^{\mu} - \mu - 2)\,v^2 + \left(-\mu^2 + e^{\mu}\mu^2\right)v - 2\,e^{\mu}\mu^2 + 2\,\mu^2}. \tag{9}$$

The resulting $R_{mm}(v)$ agrees with $\cos(v)$ on value, first and second derivative in $v = 0$ and on value only in $v = i\mu$ and $v = -i\mu$.

### *Mixed-type, mixed-order, multi-parameter EF*

When $m$ is large enough, any hybrid form is possible. It is possible to construct a method with a stability function that is an approximation to $\cos(v)$ in $v = v_k$ of order $n_k$, $k = 1 \dots r$ if $\sum_k n_k \leq m$. The conditions to impose on the

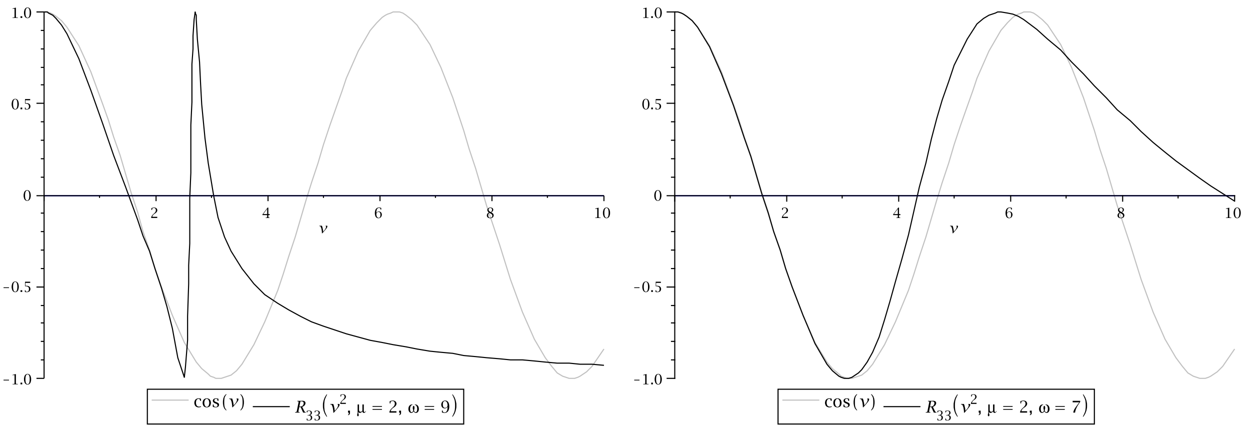


**FIGURE 1.** Plots of $R_{33}(v, \mu = 2, \omega = 9)$ (left) and $R_{33}(v, \mu = 2, \omega = 7)$ (right). The stability function is a first order approximation to $\cos(v)$ in $v = 0$, $v = 2$ and of zeroth order in $v = 9$ and $v = 7$ respectively.

stability function are

$$\begin{cases} S_{mm}^{(j)}(w_k) = e^{w_k}, & j = 0 \dots n_j \\ S_{mm}^{(j)}(-w_k) = e^{-w_k}, & j = 0 \dots n_j. \end{cases}$$

with $w_k = iv_k$, for all $v_k$ under consideration.

As an example, we have constructed an Obrechkoff method with fitting space $\{1, t, e^{\pm \mu t}, te^{\pm \mu t}, e^{\pm \omega t}\}$. From the above statement, this requires that $m = 3$. Figure 1 shows two plots: on the left $R_{33}(v, \mu = 2, \omega = 9)$ and on the right $R_{33}(v, \mu = 2, \omega = 7)$, both along the real axis. The plots show that the stability function has the same value and derivative as $\cos(v)$ in $v = 0$ and $v = 2$, while there is only an agreement on value in $v = 9$ (left) and $v = 7$ (right).

# CONCLUSION

Based on the results for single-parameter exponential fitting, we derived the necessary conditions to construct the stability function of P-stable Obrechkoff methods of arbitrary order with multi-parameter, multi-order fitting spaces. The resulting rational functions can be interpreted as interpolations of $\cos(v)$ through different points $\{v_1, \dots, v_m\}$ with multiplicities $\{n_1, \dots, n_m\}$.

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